**Problem 1 Bivariate normal distribution**

Assume that $\boldsymbol{X}$ is a bivariate normal random variable with

$$\boldsymbol{\mu} = E\boldsymbol{X} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \text{and} \quad \Sigma = \operatorname{Cov} \boldsymbol{X} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Let

$$\boldsymbol{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \boldsymbol{X}.$$

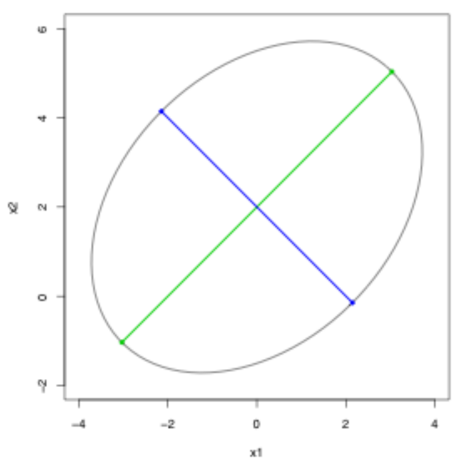
**a)** Find the mean vector and covariance matrix of $\boldsymbol{Y}$. What is the distribution of $\boldsymbol{Y}$? Are $Y_1$ and $Y_2$ independent random variables?

Let $f$ be the pdf of $\boldsymbol{X}$. Contours of $f$ are the $\boldsymbol{x}$ satisfying $f(\boldsymbol{x}) = a$ for some constant $a > 0$, or equivalently $(\boldsymbol{x} - \boldsymbol{\mu})^{\mathrm{T}} \Sigma^{-1} (\boldsymbol{x} - \boldsymbol{\mu}) = b$ for a corresponding constant $b > 0$. In the figure below, the contour for $b = 4.6$ is shown.

**b)** Explain the connections between the covariance matrix $\Sigma$, the chosen value of $b$ and features of the ellipse (e.g. principal axes and their half-lengths). Mark these features on the figure (make a drawing or use the printed figure). What is the probability that $\boldsymbol{X}$ falls within the given ellipse?

The following information might be useful:

```
> sigma=matrix(c(3,1,1,3),ncol=2)
> sigma
     [,1] [,2]
[1,]    3    1
[2,]    1    3
> eigen(sigma)
$values
[1] 4 2
$vectors
          [,1]       [,2]
[1,] 0.7071068 -0.7071068
[2,] 0.7071068  0.7071068
> qchisq(0.9,2)
[1] 4.60517
```



**Problem 2 Distributional results for $\bar{X}$ and $S^2$ for a univariate normal sample**

Let $X_1, X_2, \ldots, X_n$ be a (univariate) random sample from some population. Define $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$.

Futher, let $\mathbf{1}$ be an $n$-dimensional vector of 1s. Then $\mathbf{1}\mathbf{1}^\mathrm{T}$ is an $n \times n$ matrix of 1s. Let $I$ be an $n \times n$ identity matrix. The matrix

$$C = I - \frac{1}{n}\mathbf{1}\mathbf{1}^\mathrm{T} = \begin{pmatrix} 1-1/n & -1/n & \cdots & -1/n \\ -1/n & 1-1/n & \cdots & -1/n \\ \vdots & \vdots & \ddots & \vdots \\ -1/n & -1/n & \cdots & 1-1/n \end{pmatrix}$$

is called the *centring matrix*. Let $\boldsymbol{X} = (X_1 \ X_2 \ \cdots \ X_n)^\mathrm{T}$.

**a)** Show that $\bar{X} = \frac{1}{n}\mathbf{1}^\mathrm{T}\boldsymbol{X}$ and that $S^2 = \frac{1}{n-1}\boldsymbol{X}^\mathrm{T}C\boldsymbol{X}$.

Hint: Start by considering the $i$th component of $C\boldsymbol{X}$, and observe that $C$ is symmetric and idempotent.

Now, assume in addition that the random sample is taken from the univariate normal distribution with mean $\mu$ and variance $\sigma^2$. In the notation of TMA4267, $\boldsymbol{X} \sim N(\mu\mathbf{1}, \sigma^2 I)$.

In your first statistics course, you were told that the $T$-statistic,

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}},$$

follows a $t$-distribution with $n-1$ degrees of freedom. This result follows from $(\bar{X}-\mu)/(\sigma/\sqrt{n}) \sim N(0,1)$ and $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$, and independence of $\bar{X}$ and $S^2$.

With your new skills on the multivariate normal distribution, you can prove independence of $\bar{X}$ and $S^2$, and $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$.

**b)** Show that $\frac{1}{n}\mathbf{1}^\mathrm{T}C = \mathbf{0}^\mathrm{T}$. What does this imply about $\frac{1}{n}\mathbf{1}^\mathrm{T}\boldsymbol{X}$ and $C\boldsymbol{X}$? How can you use this to conclude that $\bar{X}$ and $S^2$ are independent?

**c)** Derive the distribution of $(n-1)S^2/\sigma^2$.

Hint: $S^2 = \frac{1}{n-1}\boldsymbol{X}^\mathrm{T}C\boldsymbol{X}$, where $C$ is symmetric and idempotent. In general, if $R$ is a symmetric and idempotent matrix with rank $r$, and $\boldsymbol{Y} \sim N(\mathbf{0}, I)$, then $\boldsymbol{Y}^\mathrm{T}R\boldsymbol{Y} \sim \chi^2_r$ (which we saw in a lecture and is stated by Fahrmeir, Kneib, Lang and Marx (2013) in Theorem B.8.2 on p. 651). Note, however, that $\boldsymbol{X}$ is not assumed to be $N(\mathbf{0}, I)$.