

## BEST METHOD TO SOLVE QUADRATIC INEQUALITIES IN ONE VARIABLE.

(By Nghi H Nguyen – Updated on July 6, 2020)

### 1. GENERALITIES

There are 3 common methods to solve quadratic inequalities. Therefore, students sometimes are confused to select the fastest and the best solving method. This article generally explains below these 3 methods and then compares them through selected examples.

Solving a quadratic inequality in one variable, in standard form  $f(x) = ax^2 + bx + c \leq 0$  (or  $\geq 0$ ), means finding all the values of  $x$  that make the inequality true.

These values of  $x$  constitute the **solution set** of the inequality.

The solution set of a quadratic inequality are expressed in the form of **intervals**.

Examples of quadratic inequalities in one variable:

$$x^2 - 8x + 7 < 9$$

$$3x^2 + 4x - 7 < 0$$

$$5x^2 - 12x \geq 17$$

$$(3x - 5)(4x + 1) < 0$$

$$3x/(x - 1) + 4x/(3 - x) > 1$$

$$1/(x - 2) + 2/(x - 3) \geq 2$$

Examples of solution set expressed in terms of intervals:

$$(-2, 3)$$

$$(-1, 2, +\infty)$$

$$(-\sqrt{2}, \sqrt{7})$$

$$[4, +\infty)$$

$$(-\infty, 7]$$

$$[-3, 12)$$

$$[3, 14]$$

$$(\sqrt{5}, \sqrt{8}]$$

### 2. STEPS IN SOLVING QUADRATIC INEQUALITIES

In general, there are 4 common steps.

**Step 1.** Transform the inequality into standard form:  $f(x) = ax^2 + bx + c < 0$  (or  $> 0$ ).

Example 1a. Solve:  $x(4x - 3) > 7$ .

In first step, transform the inequality into standard form:  $f(x) = 4x^2 - 3x - 7 > 0$ .

Example 1b. Solve:  $(2x + 5)(x - 1) < -3$

In first step, transform the inequality into standard form:  $f(x) = 2x^2 + 3x - 2 < 0$ .

**Step 2.** Solve the quadratic equation  $f(x) = 0$ . You may use one of the 8 existing common methods: factoring FOIL method, completing the square, quadratic formula, graphing, the new Diagonal Sum Method, the Bluma Method, the popular factoring AC Method, or the most recent, the “New Transforming Method” (Google, Bing, or Yahoo Search).

Before starting, find out if the equation has 2 real roots. How? Roughly estimate the Discriminant  $D$ 's value ( $D = b^2 - 4ac$ ). You don't need to calculate its exact value, unless you want to use the quadratic formula. Just use mental math to see if  $D$  is positive ( $> 0$ ) or negative ( $< 0$ ).

If  $D < 0$ , or  $D = 0$ , solve the inequality by referring to the first part of the Theorem explained in next paragraph.

If  $D > 0$ , solve the equation  $f(x) = 0$  to get the 2 real roots  $x_1$  and  $x_2$ . You may use any solving method

**Step 3.** Solve the given quadratic inequality  $f(x) < 0$  (or  $> 0$ ), based on the 2 values  $x_1$  and  $x_2$ , found in Step 2. You may choose one of the 3 common methods to solve quadratic inequalities described below.

**Step 4.** Express the solution set of the quadratic inequality in terms of **intervals**. You must know how to correctly use the interval symbols. Examples:

(2, 7): **open** interval between. The 2 end points are not included in the solution set.  
[-3, 5]: **closed** interval. The 2 end points (-3) and (5) are included in the solution set.  
[2,  $+\infty$ ): **half-closed** interval; only the end point (2) is included in the solution set  
 $(-\infty, -1]$ : **half closed** interval; only the end point (-1) is included in the solution set.

### 3. COMMON METHODS TO SOLVE QUADRATIC INEQUALITIES

There are 3 most common methods. Therefore, students are sometimes confused to select the best solving method.

#### 1. The number line and test point method.

Given a quadratic inequality in standard form  $f(x) = ax^2 + bx + c < 0$  (or  $> 0$ ), with  $a \neq 0$

Suppose the Discriminant  $D > 0$ , and the given quadratic equation has 2 real roots  $x_1$  and  $x_2$ . Plot them on a number line. They divide the number line into one **segment**  $(x_1, x_2)$  and 2 **rays**. The solution set of the quadratic equation should be either the segment, or the 2 rays. Always use the **origin O** as test point. Substitute  $x = 0$  into the inequality. If it is true, then the origin O is located on the true segment (or the true ray). If one ray is a part of the solution set, then the other ray also belongs to the solution set, due to the symmetrical property of the parabola graph.

**Note.** When  $D < 0$ , there are no real roots, meaning the graph doesn't intersect the  $ox$  axis. The number line method can't be used. In this case you must solve the inequality by the algebraic method.

When  $D = 0$ , there is a double root at  $x = -b/2a$ , this number line method can't be used. You must apply the algebraic method.

**Examples of solving quadratic inequalities by the test point method.**

Example 2. Solve:  $x^2 - 15x < 16$ .

Solution. First step, write the inequality in standard form  $f(x) = x^2 - 15x - 16 < 0$ . Second step, solve  $f(x) = 0$ . Use the Transforming Method. The 2 real roots are  $(-1)$  and  $(16)$ .

Third step, solve the inequality  $f(x) < 0$ . Plot the 2 real roots  $(-1)$  and  $(16)$  on a number-line.



The origin O is located inside the segment  $(-1, 16)$ . Use the origin O as test point. Substitute  $x = 0$  into the inequality. We get  $-16 < 0$ . It is true, then the origin O is located on the true segment.

Step 4, express the solution set in the form of open interval  $(-1, 16)$ . The 2 endpoints  $-1$  and  $16$  are not included in the solution set.

Example 3. Solve:  $f(x) = -3x^2 < -8x + 5$

Solution. First step, write  $f(x)$  in standard form:  $f(x) = -3x^2 + 8x - 5 < 0$ .

Second step, solve  $f(x) = 0$ . This is the TIP 1 ( $a + b + c = 0$ ) case.

The 2 real roots are 1 and  $5/3$ .

Third step, plot these values on a number line. The origin O is located on the left ray. Use O as test point. Substitute  $x = 0$  into the inequality. We get:  $-5 < 0$ . It is true, then the origin O is located on the true ray. By symmetry, the other ray also belongs to the solution set.

Last step, express the solution set in the form of intervals:  $(-\infty, 1)$  and  $(5/3, +\infty)$ . The 2 end points 1 and  $5/3$  are not included.

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Example 4. Solve:  $9x^2 \leq 12x - 1$

Solution. First step:  $f(x) = 9x^2 - 12x + 1 \leq 0$ .

Second step: Solve  $f(x) = 0$ . The new Transforming Method fails to solve it, this equation can't be factored. We must use the quadratic formula. The 2 real roots are  $x_1 = (2 - 1.73)/3 = 0.09$  and  $x_2 = (2 + 1.73)/3 = 1.24$ .

Third step: Plot the numbers on a number line. The origin is located on the left ray. Substitute  $x = 0$  into the inequality. We have:  $1 < 0$ . It is not true. The origin O is not on the true ray. The solution set is the segment  $(0.09, 1.24)$ .

Step 4, the solution set is the closed interval  $[0.09, 1.24]$ ; the end points are included in the solution set.

## 2. The algebraic method.

This solving method is popular in France. It is based on a theorem about the sign status of a trinomial  $f(x) = ax^2 + bx + c$ , with  $a \neq 0$ , and  $D = b^2 - 4ac$

### Theorem on the sign status of a trinomial $f(x)$ .

a. If  $D < 0$ ,  $f(x)$  has the same sign as  $a$  regardless of the values of  $x$ .

Example 5. The trinomial  $f(x) = 3x^2 - x + 7$  has  $D = b^2 - 4ac = 1 - 84 = -83 < 0$ .

This trinomial  $f(x)$  is always positive, same sign as  $a = 3$ , regardless of the values of  $x$ .

Example 6. The trinomial  $f(x) = -5x^2 + 3x - 8$  has  $D = 9 - 160 = -151 < 0$ . This  $f(x)$  is always negative ( $< 0$ ), same sign as  $a = -5$ , regardless of the values of  $x$ .

- b. When  $D = 0$ ,  $f(x)$  has the same sign as  $a$  for any values of  $x \neq (-b/2a)$ .
- c. When  $D > 0$ ,  $f(x)$  has the opposite sign of  $a$  between the 2 real roots  $x_1$  and  $x_2$ , and  $f(x)$  has the same sign as  $a$  outside the interval  $(x_1, x_2)$ .

Example 7. The trinomial  $f(x) = x^2 - 8x - 9$  has  $D = 64 + 36 = 100 = 10^2 > 0$ . The equation  $f(x) = 0$  has 2 real roots  $(-1)$  and  $(9)$ . The trinomial  $f(x)$  is negative ( $< 0$ ) within the interval  $(-1, 9)$ , and  $f(x)$  is positive ( $> 0$ ) outside this interval.

Example 8. The trinomial  $f(x) = -x^2 + 5x - 4 = 0$  has  $D = 25 - 16 = 9 = 3^2 > 0$ . The equation  $f(x) = 0$  has 2 real roots  $1$  and  $4$ . The trinomial  $f(x)$  is positive, opposite to the sign of  $a = -1$ , inside the interval  $(1, 4)$ .

### The Theorem's development

- a. **When  $D < 0$ .** We can write  $f(x)$  in the form  $f(x) = a*(x^2 + bx/a + c/a)$ . **(1)**

Recall the equation developed to find the quadratic formula:

$$(x^2 + bx/a + cx/a) = 0$$

$$x^2 + bx/a + b^2/4a^2 - b^2/4a^2 + c/a = 0$$

$$(x + b/2a)^2 - (b^2 - 4ac)/4a^2 = 0 \quad (\text{Call } D = b^2 - 4ac).$$

$$(x + b/2a)^2 - D/4a^2 = 0$$

Substitute this relation into the equation **(1)**, we get:

$$f(x) = a*[(x + b/2a)^2 - D/4a^2] \quad \mathbf{(2)}$$

**When  $D < 0$ ,** the quantity inside the parenthesis is always positive. Therefore,  $f(x)$  has the same sign as  $a$  regardless of the values of  $x$ .

- a. **When  $D = 0$ ,** the equation (2) reduces to:  $f(x) = a*(x + b/2a)^2$ .  
We see that  $f(x)$  has the same sign as  $a$  regardless of the values of  $x$ , since the quantity  $(x + b/2a)^2$  is always positive.
- b. **When  $D > 0$ .** There are 2 real roots  $x_1$  and  $x_2$ , with their sum  $(x_1 + x_2) = -b/a$ , and their product  $(x_1*x_2) = c/a$ .  
We can write  $f(x)$  in the form  $f(x) = a*(x^2 + bx/a + c/a)$ . The quantity in parenthesis is a quadratic equation that can be factored into 2 binomials in  $x$  with  $x_1$  and  $x_2$  as real roots.

$$f(x) = a(x - x_1)(x - x_2) \quad (1)$$

Now, setup a **Sign Chart** (sign table) to study the sign status of  $f(x)$  when  $x$  varies from  $-\infty$  to  $+\infty$  and passes by the 2 values  $x_1$  and  $x_2$ . Suppose  $x_2 > x_1$ .

$x$		$-\infty$		$x_1$		$x_2$		$+\infty$
$x - x_1$		-		0	+			+
$x - x_2$		-			-	0		+
$(x - x_1)(x - x_2)$		+		0	-	0		+

We see from the relation (1) and the last line of the chart that  $f(x)$  has the opposite sign of  $a$  between the 2 real roots  $x_1$  and  $x_2$ . Outside the interval  $(x_1, x_2)$ ,  $f(x)$  have the same sign as  $a$ .

### Graphic interpretation of the Theorem.

We can easily understand this Theorem by looking at the parabola graph of a quadratic equation.

**If  $a > 0$** , the parabola opens **upward**. Between the two  $x$ -intercepts (real roots) of the parabola, a part of the parabola is **below** the  $x$ -axis, meaning  $f(x)$  is negative ( $< 0$ ) inside this interval  $(x_1, x_2)$ , opposite in sign to  $a$  (positive).

**If  $a < 0$** , the parabola opens **downward**. Between the two  $x$ -intercepts (real roots), a part of the parabola is **above** the  $x$ -axis, meaning  $f(x)$  is positive inside this interval  $(x_1, x_2)$ , opposite in sign to  $a$  (negative).

Example 9. The graph of the trinomial  $f(x) = x^2 - 7x - 8$  is an upward parabola whose  $x$ -intercepts are at  $-1$  and  $8$ . Within the interval  $(-1, 8)$  a part of the parabola is below the  $x$ -axis, meaning  $f(x)$  is negative ( $< 0$ ) inside this interval, opposite in sign to  $a$  (positive)

Example 10. The graph of  $f(x) = -x^2 + 11x - 10$  is a downward parabola whose  $x$ -intercepts are at  $1$  and  $10$ . Within the interval  $(1, 10)$ , a part of the parabola is above the  $x$ -axis, meaning  $f(x)$  is positive ( $> 0$ ) in this interval, opposite in sign to  $a$

**NOTE.**

Students study once the theorem's development, then they apply it to solve various types of quadratic inequalities.

**3. The graphing method**

The concept of this method is simple. When the parabola graph of the trinomial  $f(x)$  is above the  $x$ -axis, the trinomial  $f(x)$  is positive, and vice-versa.

You don't need to accurately graph the parabola. Based on the 2 real roots obtained by solving  $f(x) = 0$ , you may just draw a simple sketch of the parabola.

Pay only attention to if the parabola opens upward or downward, by considering the sign of  $a$ . Finally, express the solution set in the form of intervals.

By this graphing method, you may solve a system of two, or even three, quadratic inequalities, by graphing them on the same coordinate grid.

**EXAMPLES OF SOLVING QUADRATIC INEQUALITIES**

Example 11. Solve:  $f(x) = 13x^2 - 7x + 8 > 0$ .

Solution. The Discriminant  $D = 49 - 416 < 0$ . There are no real roots. Use the algebraic method (Theorem) to solve it. Since  $a$  is positive,  $f(x)$  is always positive ( $> 0$ ) regardless of the values of  $x$ . The inequality is always true.

Example 12. Solve:  $f(x) = -5x^2 + 6x - 10 > 0$ .

Solution.  $D = 36 - 200 < 0$ . There are no real roots. Use the Theorem to solve. The trinomial  $f(x)$  is always negative ( $< 0$ ), same sign as  $a = -5$ , regardless of  $x$ . The equation is always not true. The graph is a downward parabola that doesn't intersect the  $x$ -axis.

Example 13. Solve:  $f(x) = 3x^2 - 7x - 10 > 0$ .

Solution by the number line and test point method.

The equation  $f(x) = 0$  has 2 real roots  $-10/3$  and  $1$ . Plot these numbers on a number line.

The origin  $O$  is located inside the segment  $(-10/3, 1)$ . Substitute  $x = 0$  into the inequality. We have:  $10 > 0$ . It is not true, then  $O$  is not on the solution set. The solution set is the 2 rays. Solution set by intervals:  $(-\infty, -10/3)$  and  $(1, +\infty)$ .

Solution by the algebraic method. The equation  $f(x)$  has 2 real roots  $-10/3$  and  $1$ . Inside the segment  $(-10/3, 1)$   $f(x)$  is negative ( $< 0$ ), opposite in sign to  $a = 5$ . Therefore the 2 rays are the solution set. Solution set: open intervals  $(-\infty, -10/3)$  and  $(1, +\infty)$ . There is no need to draw the number line and to do the test point method.

Example 14. Solve:  $f(x) = -3x^2 + 18x - 15 < 0$ .

Solution by the number line. The equation  $f(x) = 0$  has 2 real roots  $1$  and  $5$ . Plot these numbers on a number line. The origin  $O$  is outside the segment  $(1, 5)$ . Substitute  $x = 0$  into the inequality. We get:  $-15 < 0$ . It is true then  $O$  is located on the true ray. The 2 rays are the solution set. Answers:  $(-\infty, 1)$  and  $(5, +\infty)$ .

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Solution by the algebraic method. The 2 real roots are  $1$  and  $5$ . Outside the interval  $(1, 5)$ ,  $f(x)$  is positive ( $> 0$ ), opposite in sign to  $a = -3$ . The 2 rays are the solution set. Answers:  $(-\infty, 1)$  and  $(5, +\infty)$ .

### IMPORTANT REMARKS AND CONCLUSION

**Important Remark 1.** When the inequality is written in the form  $f(x) \leq 0$  (or  $\geq 0$ ), the 2 end (critical) points are automatically included in the solution set.

**Important Remark 2.** When  $D < 0$  and  $D = 0$ , the number line and test point method can't be used. The algebraic method (Theorem) must be applied.

**Important Remark 3.** When  $D > 0$ , there are 2 real roots. Plot these roots on the number line, they divide it into 3 areas. Taking the origin  $O$  as test point in one area is enough. It is useless to have 3 test points for 3 areas. Using 3 test points for 3 areas is effective for first-time lesson explaining, but it is useless when students have mastered the solving process. During tests/exams, time is limited. Students must find answers as fast as possible. Taking 3 test points would cost too much precious time.

**Remark 4.** There is another algebraic method that discusses the sign of the 2 binomials  $(x - x_1)$  and  $(x - x_2)$ . This method is unadvised because it takes too much time and it easily leads to error/mistakes. You'd better use the algebraic method that is much simpler and faster.

Example 15. Solve:  $f(x) = 9x^2 - 8x - 17 < 0$ .

Solution. The equation  $f(x) = 0$  has 2 real roots  $(-1)$  and  $(17/9)$ . Create a Sign Chart with  $x$  varies from  $(-\infty)$  to  $(+\infty)$  and passes by the 2 real roots.

$x$	$-\infty$	$-1$	$17/9$	$+\infty$
$x + 1$	-	0	+	+
$x - 17/9$	-		-	+
$f(x)$	+	0	-	+

The solution set of  $f(x) < 0$  is the open interval  $(-1, 17/9)$ .

**Conclusion.**

The **algebraic method** may be the **best** one to solve quadratic inequalities. It is fast because it doesn't require to draw the number line and to take the test points each time. Just by considering the **sign** of **a**, students may immediately know in which intervals **f(x)** would be **positive** (or negative).

In addition, it helps solving quadratic inequalities when **D < 0** or when **D = 0**.

For multiple choice answers in **tests/exams**, this method may be the **best** one.

[This article was written by Nghi H. Nguyen, author of the new Transforming Method (Google, Yahoo, or Bing Search) to solve quadratic equations – Updated on May 6, 2020]